Quality Factor

The general meaning of the quality factor Q of a circuit is the ratio between the energy stored in the circuit (in C and L) and the energy dissipated (by R):

$$Q = 2\pi \frac{\text{maximum energy stored}}{\text{energy dissipated per cycle}}$$

We can show that this definition is the same as the Q for the series RCL circuit. The energy is stored in the inductor and the capacitor alternatively, and the maximum energy stored in L is the same as that in C.

The energy store in L is:

$$W_L = \int_0^T v(t)\ i(t)dt = \int_0^T i(t)L\frac{di(t)}{dt}dt = L\int_0^{I_p} i di = \frac{1}{2}LI_p^2 = LI_{rms}^2$$

where $i(T) = I_p = \sqrt{2}I_{rms}$ is the maximum or peak current through L.

The energy dissipated in R per cycle $T_0 = 2\pi/\omega_0 = 1/f_0$ is:

$$W_R = P_R T_0 = I_{rms}^2 R T_0$$

Then

$$2\pi\frac{LI_{rms}^2}{I_{rmw}^2 RT_0} = 2\pi f_0 \frac{L}{R} = \frac{\omega_0 L}{R} = Q$$

We can also use the energy store in C:

$$W_C = \int_0^T v(t)\ i(t)dt = \int_0^T v(t)C\frac{dv(t)}{dt}dt = C\int_0^{V_p} v dv = \frac{1}{2}CV_p^2 = CV_{rms}^2$$

where $v(T) = V_p = \sqrt{2}V_{rms}$ is the maximum or peak voltage across C, and

$V_{rms} = I_{rms}/\omega_0 C$. Then

$$2\pi \frac{CV_{rms}^2}{I_{rms}^2 RT_0} = 2\pi f_0 \frac{CI_{rms}^2/\omega_0^2 C^2}{I_{rms}^2 R} = \frac{1}{\omega_0 CR} = Q$$

Peak Frequency and Bandwidth

For a series RCL circuit with voltage input, the impedance is

$$\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1} = -\frac{1}{Q}, \quad \frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2} = \frac{1}{Q}$$

and at resonance when $\omega = \omega_0 = 1/\sqrt{LC}$, we have

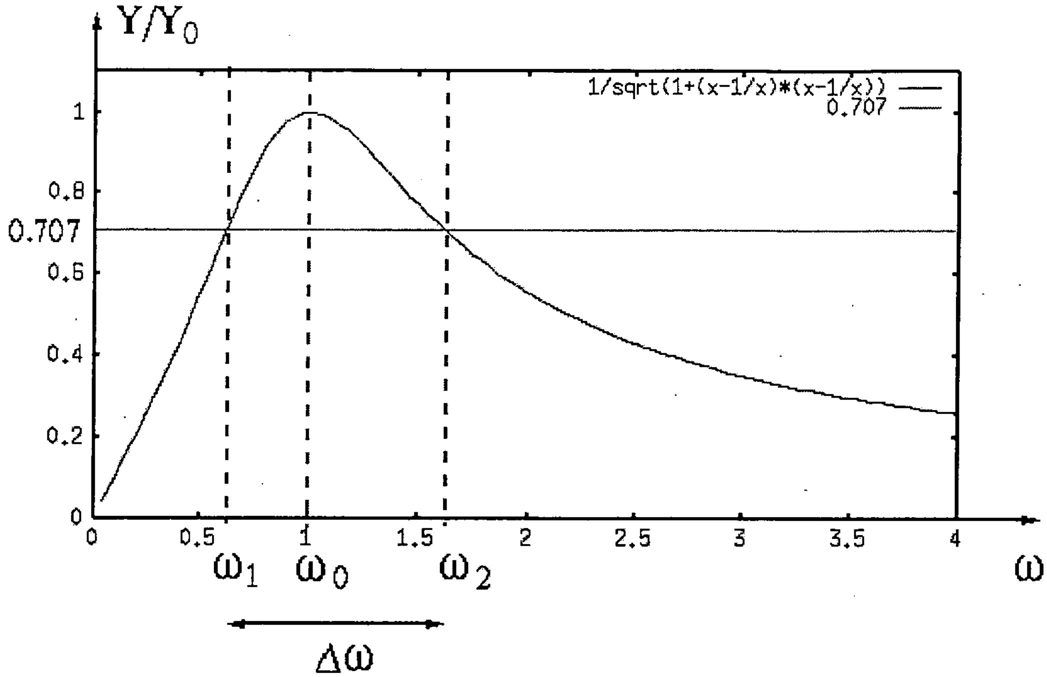
$$Z_0 = R = \frac{1}{Y_0}$$

Consider the ratio

$$\frac{Y}{Y_0} = \frac{Z_0}{Z} = \frac{R}{R + j(\omega L - 1/\omega C)} = [1 + j(\frac{\omega L}{R} - \frac{1}{\omega CR})]^{-1}$$

As $Q = \omega_0 L/R = 1/\omega_0 CR$, we substitute $L/R = Q/\omega_0$ and $1/RC = Q\omega_0$ into the equation above and get

$$\frac{Y}{Y_0} = [1 + jQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})]^{-1}$$



At the resonant frequency $\omega = \omega_0$, the ratio $Y/Y_0 = 1$ reaches the peak, and when ω is either lower or higher than ω_0, the magnitude ratio Y/Y_0 is smaller. The bandwidth is defined as

$$\Delta\omega \triangleq \omega_2 - \omega_1$$

where $\omega_2 > \omega_1$ are the two cut-off frequencies (or half-power frequency) at which $|Y/Y_0| = 1/\sqrt{2} = 0.707$ (the power is halved). This requires

$$\frac{Y}{Y_0} = \frac{1}{1 \pm j1}, \quad \text{i.e.} \quad \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = \pm\frac{1}{Q}$$

The two cut-off frequencies should satisfy

$$\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1} = -\frac{1}{Q}, \quad \frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2} = \frac{1}{Q}$$

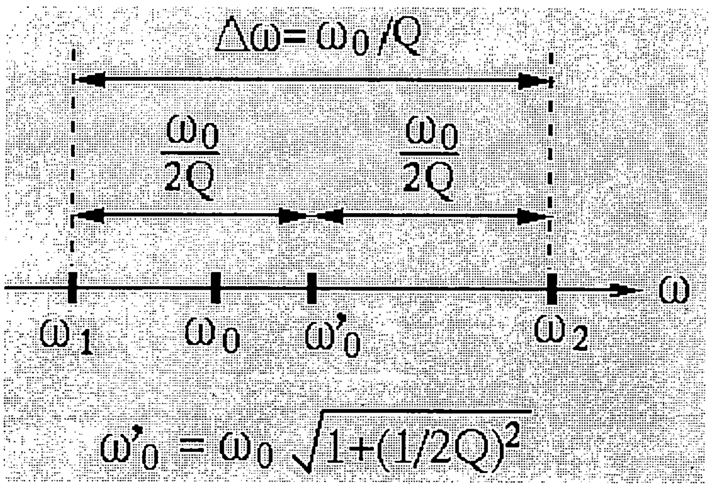
Solving these two equations, we get:

$$\omega_1 = \omega_0[\sqrt{1 + (\frac{1}{2Q})^2} - \frac{1}{2Q}] = \omega_0(\sqrt{1+\zeta^2} - \zeta), \quad \omega_2 = \omega_0[\sqrt{1 + (\frac{1}{2Q})^2} + \frac{1}{2Q}] = \omega_0(\sqrt{}$$

from which we can find the bandwidth

$$\Delta\omega = \omega_2 - \omega_1 = \frac{\omega_0}{Q} = 2\zeta\omega_0, \quad \text{or} \quad \Delta f = f_2 - f_1 = \frac{f_0}{Q} = 2\zeta f_0$$

i.e., the bandwidth is inversely proportional to the quality factor Q. Also note that the middle point between ω_1 and ω_2 is $\omega_0' = (\omega_1 + \omega_2)/2 = \omega_0\sqrt{1+\zeta^2} > \omega_0$, i.e., $\omega_2 - \omega_0 > \omega_0 - \omega_1$.



In practice, $Q = 1/2\zeta$ is usually much greater than 1 (typically $Q > 10$, i.e., $\zeta < 0.05$), we have $\sqrt{1+(1/2Q)^2} = \sqrt{1+\zeta^2} \approx 1$ and

$$\omega_1 = \omega_0 - \frac{\omega_0}{2Q} = \omega_0(1-\zeta), \qquad \omega_2 = \omega_0 + \frac{\omega_0}{2Q} = \omega_0(1+\zeta)$$

we therefore get these simple relations:

$$\omega_2 - \omega_0 = \omega_0 - \omega_1\omega_0\zeta, \qquad \Delta\omega = \omega_2 - \omega_1 = \frac{\omega_0}{Q} = 2\zeta\omega_0$$

For a parallel RCL circuit with current input, due to the duality between current and voltage, parallel and series configuration, the exactly same derivation of bandwidth can be carried out as for the RCL series circuit with voltage input, and the same conclusions can be obtained.

Summary:

- The resonant frequency of both series and parallel RCL circuits is completely determined by L

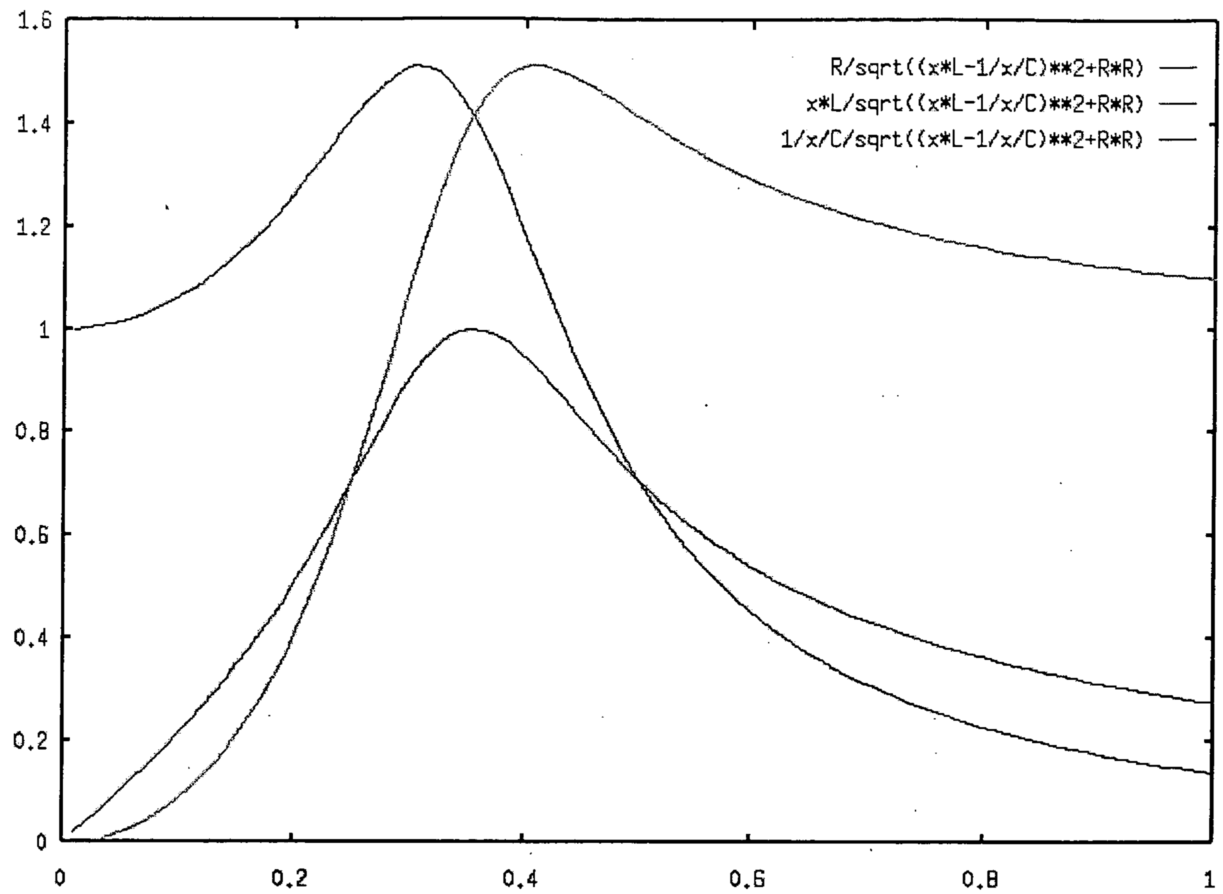
and C: $\omega_0 = 1/\sqrt{LC}$, independent of the resistance R in the circuit.

- At resonant frequency, the impedance $Z = 1/Y$ of a series RCL circuit is real and reaches minimum and the current through the three components caused by a voltage input reaches maximum; the admittance $Y = 1/Z$ of a parallel RCL circuit is real and reaches minimum and the voltage across the three components caused by a current source reaches maximum.
- In series RCL with voltage input and parallel RCL with current input, the quality factor Q is proportional to the ratio between L and C:

$$Q_s = \frac{1}{R}\sqrt{\frac{L}{C}}, \quad Q_p = R\sqrt{\frac{C}{L}}$$

The resistance R plays a different role: in series RCL, Q_s is inversely proportional R (the larger R, the smaller Q_s, the more energy lost and the wider bandwidth), while in parallel RCL, Q_p is proportional to R (the larger R, the larger Q_p, the less energy lost and the narrower bandwidth).

- At resonant frequency, the impedance of a series RCL circuit reaches minimum, consequently the current $\mathbf{I}$ reseaches maximum and so does the voltage across the resistor $\mathbf{V_R} = \mathbf{I}R$. However, the voltage across the inductor $\mathbf{V_L} = \mathbf{I}Z_L$ reaches maximum at a frequency slightly higher than the resonant frequency as $Z_L = j\omega L$ is proportional to ω, and the voltage across the capacitor $\mathbf{V_C} = \mathbf{I}Z_C$ reaches maximum at a frequency slightly lower than the resonant frequency as $Z_C = 1/j\omega C$ is inversely proportional to ω.



Example 1:

A series RCL circuit composed of an inductor $L = 80\mu H$ and $R = 8\Omega$ and a capacitor C is connected to a voltage source. Find the value of C for this circuit to resonate at $f = 400\ kHz$, also find the bandwidth.

$$\omega_0 = \sqrt{\frac{1}{LC}}, \quad C = \frac{1}{\omega_0^2 L} = \frac{1}{(2\pi 400 \times 10^3)^2 \times 80 \times 10^{-6}} = 20nF$$

The quality factor is

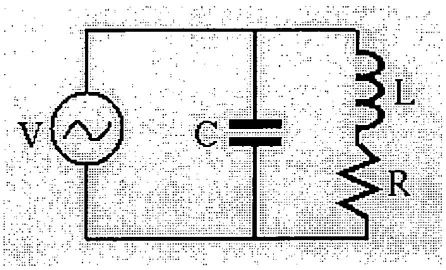
$$Q = \frac{\omega_0 L}{R} = \frac{2\pi 400 \times 10^3 \times 80 \times 10^{-6}}{8} = 25$$

The bandwidth is

$$f_2 - f_1 = \frac{f_0}{Q} = \frac{400 \times 10^3}{25} = 16\ kHz$$

Example 2:

In reality, all inductors have a non-zero resistance, therefore a parallel resonance circuit should be modeled as shown in the figure.



The admittance is:

$$Y(\omega) = \frac{1}{R + j\omega L} + j\omega C = \frac{R - j\omega L + j\omega C(R^2 + \omega^2 L^2)}{R^2 + \omega^2 L^2}$$

As frequency ω appears in the real part $Re[Y(\omega)]$ as well as in the imagineary part $Im[Y(\omega)]$, the resonant frequency that minimizes $|Y(\omega)|$ has to be found by

$$\frac{d}{d\omega}|Y(\omega)| = 0$$

However, when the quality factor $Q = \omega_0 L/R$ associated with the non-ideal inductor is large enough ($Q > 20$), all previous discussed relations for ideal inductors still hold, and the resonant frequency ω_0 can still be found approximately by the previous approach of letting $Im[Y(\omega)] = 0$:

$$\omega_0 L = \omega_0 C(R^2 + \omega_0^2 L^2), \quad \Longrightarrow \quad \omega_0 = \sqrt{\frac{1}{LC} - (\frac{R}{L})^2}$$

For ω_0 to be real, we must have

$$\frac{1}{LC} > (\frac{R}{L})^2, \quad \text{i,e,} \quad R < \sqrt{\frac{L}{C}}$$

Typically we have $R \ll \sqrt{L/C}$, and the resonant frequency is

$$\omega_0 = \sqrt{\frac{1}{LC} - (\frac{R}{L})^2} \approx \frac{1}{\sqrt{LC}}$$

Note: For the same reason, when considering the transfer function of a series RCL circuit when the output is the voltage across either C or L, the peak frequency ω_p is not exactly the same as the resonant frequency ω_0, which only minimizes the denominator, but the numerator is still a function of ω. Only when the output is the voltage across R (i.e., the numerator is R, no longer a function of ω), will the resonant frequency ω_0 be the same as the peak frequency.

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Next: Sinusoidal AC Power **Up:** Chapter 3: AC Circuit **Previous:** Series and Parallel Resonance

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